

# A new class of time-dependent Bell inequalities in Wigner form

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## Abstract

We derive a new class of time-dependent Bell inequalities in Wigner form under the assumption of locality in the framework of Kolmogorov's probability theory. We consider violation of the obtained inequalities for three cases: spin correlations in an external magnetic field; oscillations of neutral pseudoscalar mesons; and decays of a pseudoscalar into a fermion-antifermion pair.

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## I. INTRODUCTION

One of the fundamental questions that appear in both non-relativistic quantum mechanics (NRQM) and quantum field theory (QFT) is how the physical properties of micro-objects are related to the measurement procedure performed with a macroscopic device. According to the Copenhagen interpretation of quantum mechanics, one can refer to properties of micro-objects only if the macroscopic measurement procedure for them is defined. The maximum amount of information related to a micro-object's properties is then defined by the number of its "characteristics" which can be measured by a given macro-device. In quantum theory these characteristics are represented by sets of commuting Hermitan operators.

However it is possible to introduce non-commuting operators corresponding to physical observables for a given micro-system, e.g. the operator of spin  $s = 1/2$  projection onto some non-parallel directions. In NRQM and QFT this operator is usually defined as  $\vec{s} = \vec{O}/2$  [42]. Cartesian projections of this operator satisfy the following commutation relations:

$$[O^i, O^j] = 2i\epsilon^{ijk}O^k, \text{ where } \epsilon^{123} = +1.$$

The above operators do not have a common system of eigenvectors, hence it is impossible to measure simultaneously any two spin projections onto non-parallel directions. Bohr's principle of complementarity is a philosophical reflection of this fact, while Heisenberg's uncertainty principle is a mathematical one.

Is it possible for characteristics of a macro-system, which are described by non-commuting operators, to be simultaneously the elements of physical reality, even in the absence of a macro-device which is able to measure all of them simultaneously? This question was first introduced in [1], and N. Bohr addressed this in [2], presenting another philosophic view on the problem of measurement in the NRQM. Fifty years ago J. Bell suggested an experimental method [3–5], which has been further developed by J. Clauser, M. Horne, A Shimony and R. Holt [6], and has been widely discussed since then [7–9].

One of the ways to express mathematically the impossibility of simultaneous measurement of a set of observables with a single macro-device is to suggest that the joint probability of existence of the given set is non-negative. This assumption has been used by J. Bell in [3], although not as an explicit statement, as was done later by E. Wigner [10]. The inequalities obtained are now known as Bell inequalities in Wigner form or just as Wigner

inequalities. Calculation or measurement of the probability in quantum mechanics is a well defined procedure, making the Bell inequalities in Wigner form more “natural” in some sense than the classic Bell inequalities, which are formulated in terms of correlators of operators corresponding to two observables. Derivation of classic Bell inequalities using the non-negative joint probabilities has been done in [11].

An attempt at a relativistic generalization of the Bell inequalities in Wigner form may be found in [12], where various relativistic corrections were considered for decay of a pseudoscalar particle into a fermion-antifermion pair. Corrections due to non-parallel momenta of fermions (e.g. due to soft photon radiation) were taken into account, as well as corrections due to finite distance to the spin analyzers. It was shown that these effects have almost negligible influence upon the classic Wigner inequalities. The time dependence, however, was not taken into account in [12].

There are well known time-dependent Leggett-Garg inequalities [13], based on the idea of macroscopic realism, i.e. that every physical observable has a defined value at every moment of time, and that the measurement performed does not affect the subsequent dynamics of the observable. Leggett-Garg inequalities use the correlation between the values of the same observable at distinct moments of time. For example, it is possible to examine the correlations between spin projections for the spin precession in an external magnetic field [14]. Experimental study of violation of Leggett-Garg inequalities is closely related to the idea of “weak measurement” [15, 16]; e.g. nanomechanical resonators [17] may be used for such tests. Various generalizations of Leggett-Garg inequalities are actively discussed nowadays [18], as well as their possible applications to particle physics [19]. However none of the Leggett-Garg inequalities are suitable for tests of Bohr’s complementarity principle; it is necessary to find a class of inequalities which combine the principle of local realism with time dependence.

Many works [20] – [31] attempt to incorporate time dependence into classic Bell/Wigner inequalities. Papers [23] – [27] introduce Bell-like inequalities for neutral pseudoscalar mesons (usually  $K$ ). Two main topics are considered: first, starting from [23], the time-independent inequalities are constructed in terms of flavour,  $CP$ -violation, or mass/lifetime eigenstates, and the time dependence arises during the calculation of the probabilities in the framework of quantum mechanics. The resulting inequalities contain  $CP$ -violation parameters  $\varepsilon$  and  $\varepsilon'$ . There are also attempts [27] to include into the inequalities additional

correlation functions which depend on time difference. In yet another approach [20] – [22], the authors derive special time-dependent inequalities based on the principles of causality and locality, which can be used to test the joint existence of two characteristics which can not simultaneously be measured. These inequalities, however, are not general and are only applicable to the oscillations of neutral mesons. Finally in [27] – [31], the time-dependent inequalities similar to [6] are introduced, but there are certain difficulties with violation of these inequalities in quantum mechanics.

The main property of all variants of Bell-like inequalities is their conservation if the joint probabilities are non-negative and the measurements are local. It is possible, however, to introduce some macroscopic configurations of measurement devices leading to violation of the inequalities. There are works with criticism of Bell inequalities per se [32, 33], stating that the derivation of the inequalities is not correct itself due to the mutual unconformity of their constituent values. To address these concerns it is necessary to formulate clearly our basic assumptions. The question of the non-locality of measurements is more or less resolved by Eberhard’s theorem [34].

Unlike [20] – [22], we do not try to abandon the concept of Bell inequalities, moreover they naturally re-appear in our analysis. On the other hand, we do not introduce additional correlation functions which change the structure of the inequalities, contrary to the approaches stimulated by [27].

## II. BELL INEQUALITIES IN WIGNER FORM FOR A SPIN-ANTICORRELATED FERMION PAIR IN AN EXTERNAL FIELD

Following the logic of works [32] we use Kolmogorov’s approach to probability.

Let a pseudoscalar particle decay at the moment  $t_0$  to a fermion-antifermion pair. Below we denote the antifermion with the index “1” and the fermion with the index “2”. Let the spin projections of the fermion and antifermion onto three non-parallel directions  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  be simultaneously the elements of physical reality. Let us define the spin 1/2 projection onto any of the axes  $\vec{n}$  as

$$s_{\vec{n}} = \pm \frac{1}{2} \equiv n_{\pm}.$$

Let the indices  $\{\alpha, \beta, \gamma\} = \{+, -\}$ . Then the spin projections at the initial moment of time  $t = t_0$  onto any direction are anticorrelated:

$$a_{\pm}^{(1)}(t_0) = -a_{\mp}^{(2)}(t_0). \quad (1)$$

Note that in QFT the above condition is automatically satisfied in the case of a strong or electromagnetic decay with  $P$ -conservation. It is easy to construct an example of a Hamiltonian which provides full anticorrelation at the initial moment of time:

$$\mathcal{H}^{(PS)}(x) = g \varphi(x) (\bar{f}(x) \gamma^5 f(x))_N, \quad (2)$$

where  $\varphi(x)$  is a pseudoscalar field and  $\bar{f}(x)$  and  $f(x)$  are fermionic fields.

Let  $\Omega$  be the set of elementary outcomes  $\omega_i$ . For each of them, the whole set of spin projections is the element of physical reality:  $\{a_{\alpha}^{(1)} b_{\beta}^{(1)} c_{\gamma}^{(1)} a_{\alpha'}^{(2)} b_{\beta'}^{(2)} c_{\gamma'}^{(2)}\}$ . This set is time-independent.

At time  $t = t_0$  let us define the events  $\mathcal{K}_{a_{\alpha}^{(1)} b_{\beta}^{(1)} c_{\gamma}^{(1)} a_{-\alpha}^{(2)} b_{-\beta}^{(2)} c_{-\gamma}^{(2)}}(t_0) \subseteq \Omega$  such that the elements of the physical reality are the sets of projections of a fermion-antifermion spin pair on three non-parallel directions, anticorrelated as (1)  $\{a_{\alpha}^{(1)} b_{\beta}^{(1)} c_{\gamma}^{(1)} a_{-\alpha}^{(2)} b_{-\beta}^{(2)} c_{-\gamma}^{(2)}\}$ . The set of such events by definition forms a  $\sigma$ -algebra  $\mathcal{F}(t_0)$ . It is possible to introduce a probability measure  $w$  on  $(\Omega, \mathcal{F})$  which is real and always non-negative. It is also additive ( $\sigma$ -additive) for non-overlapping events. Then it is possible to prove the inequality

$$w(a_+^{(2)}, b_+^{(1)}, t_0) \leq w(c_+^{(2)}, b_+^{(1)}, t_0) + w(a_+^{(2)}, c_+^{(1)}, t_0). \quad (3)$$

If one drops the  $t_0$  in (3), it becomes identical to the well-known inequality

$$w(a_+^{(2)}, b_+^{(1)}) \leq w(c_+^{(2)}, b_+^{(1)}) + w(a_+^{(2)}, c_+^{(1)}). \quad (4)$$

To prove (3) and, consequently (4), it is necessary to consider events

$$\begin{aligned} \mathcal{A}(t_0) &= \mathcal{K}_{a_{-}^{(1)} b_{+}^{(1)} c_{+}^{(1)} a_{+}^{(2)} b_{-}^{(2)} c_{-}^{(2)}}(t_0) \cup \mathcal{K}_{a_{-}^{(1)} b_{+}^{(1)} c_{-}^{(1)} a_{+}^{(2)} b_{-}^{(2)} c_{+}^{(2)}}(t_0), \\ \mathcal{B}(t_0) &= \mathcal{K}_{a_{-}^{(1)} b_{+}^{(1)} c_{-}^{(1)} a_{+}^{(2)} b_{-}^{(2)} c_{+}^{(2)}}(t_0) \cup \mathcal{K}_{a_{+}^{(1)} b_{+}^{(1)} c_{-}^{(1)} a_{-}^{(2)} b_{-}^{(2)} c_{+}^{(2)}}(t_0), \\ \mathcal{C}(t_0) &= \mathcal{K}_{a_{-}^{(1)} b_{+}^{(1)} c_{+}^{(1)} a_{+}^{(2)} b_{-}^{(2)} c_{-}^{(2)}}(t_0) \cup \mathcal{K}_{a_{-}^{(1)} b_{-}^{(1)} c_{+}^{(1)} a_{+}^{(2)} b_{+}^{(2)} c_{-}^{(2)}}(t_0), \end{aligned}$$

belonging to the  $\sigma$ -algebra  $\mathcal{F}(t_0)$ . Among all the indices  $\{a_{\alpha}^{(1)} b_{\beta}^{(1)} c_{\gamma}^{(1)} a_{-\alpha}^{(2)} b_{-\beta}^{(2)} c_{-\gamma}^{(2)}\}$ , it is possible to fix not six but only three, which relate to the spin projection of any fermion to each of

three axes. Then

$$\begin{aligned}
w\left(a_+^{(2)}, b_+^{(1)}, t_0\right) &= \sum_{\omega_i \in \mathcal{A}(t_0)} \left( w\left(a_+^{(2)}, b_+^{(1)}, c_+^{(2)}, \omega_i\right) + w\left(a_+^{(2)}, b_+^{(1)}, c_-^{(2)}, \omega_i\right) \right); \\
w\left(c_+^{(2)}, b_+^{(1)}, t_0\right) &= \sum_{\omega_j \in \mathcal{B}(t_0)} \left( w\left(a_+^{(2)}, b_+^{(1)}, c_+^{(2)}, \omega_j\right) + w\left(a_-^{(2)}, b_+^{(1)}, c_+^{(2)}, \omega_j\right) \right); \\
w\left(a_+^{(2)}, c_+^{(1)}, t_0\right) &= \sum_{\omega_k \in \mathcal{C}(t_0)} \left( w\left(a_+^{(2)}, b_+^{(1)}, c_-^{(2)}, \omega_k\right) + w\left(a_+^{(2)}, b_-^{(1)}, c_-^{(2)}, \omega_k\right) \right).
\end{aligned}$$

The sum  $w\left(c_+^{(2)}, b_+^{(1)}, t_0\right) + w\left(a_+^{(2)}, c_+^{(1)}, t_0\right)$  is defined on the set

$$\begin{aligned}
\mathcal{B}(t_0) \cup \mathcal{C}(t_0) &= \left( \mathcal{K}_{a_-^{(1)} b_+^{(1)} c_-^{(1)} a_+^{(2)} b_-^{(2)} c_+^{(2)}}(t_0) \cup \mathcal{K}_{a_+^{(1)} b_+^{(1)} c_-^{(1)} a_-^{(2)} b_-^{(2)} c_+^{(2)}}(t_0) \right) \cup \\
&\cup \left( \mathcal{K}_{a_-^{(1)} b_+^{(1)} c_+^{(1)} a_+^{(2)} b_-^{(2)} c_-^{(2)}}(t_0) \cup \mathcal{K}_{a_-^{(1)} b_-^{(1)} c_+^{(1)} a_+^{(2)} b_+^{(2)} c_-^{(2)}}(t_0) \right),
\end{aligned}$$

a subset of which is the event  $\mathcal{A}(t_0)$ . Then due to the non-negativity of probabilities, (3) is proven.

By changing the directions of the axes  $\vec{a}$  and  $\vec{b}$  to their opposite, as was done in [12], it is possible to obtain three more inequalities, analogous to (3):

$$\begin{aligned}
w\left(a_+^{(2)}, b_-^{(1)}, t_0\right) &\leq w\left(c_+^{(2)}, b_-^{(1)}, t_0\right) + w\left(a_+^{(2)}, c_+^{(1)}, t_0\right); \\
w\left(a_-^{(2)}, b_+^{(1)}, t_0\right) &\leq w\left(c_+^{(2)}, b_+^{(1)}, t_0\right) + w\left(a_-^{(2)}, c_+^{(1)}, t_0\right); \\
w\left(a_-^{(2)}, b_-^{(1)}, t_0\right) &\leq w\left(c_+^{(2)}, b_-^{(1)}, t_0\right) + w\left(a_-^{(2)}, c_+^{(1)}, t_0\right).
\end{aligned} \tag{5}$$

After the time interval  $\Delta t = t - t_0$ , let the fermion and antifermion become spatially well separated. Then, under the assumption of locality, one can write

$$\begin{aligned}
w\left(a_+^{(2)}, b_+^{(1)}, t\right) &= w\left(a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t)\right) w\left(b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t)\right) w\left(a_+^{(2)}, b_+^{(1)}, t_0\right) + \\
&+ w\left(a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t)\right) w\left(b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t)\right) w\left(a_-^{(2)}, b_+^{(1)}, t_0\right) + \\
&+ w\left(a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t)\right) w\left(b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t)\right) w\left(a_+^{(2)}, b_-^{(1)}, t_0\right) + \\
&+ w\left(a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t)\right) w\left(b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t)\right) w\left(a_-^{(2)}, b_-^{(1)}, t_0\right).
\end{aligned}$$

Using the inequalities (3) and (5) one can obtain the following inequality:

$$\begin{aligned}
&w\left(a_+^{(2)}, b_+^{(1)}, t\right) \leq \\
&\leq w\left(a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t)\right) w\left(b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t)\right) \left( w\left(c_+^{(2)}, b_+^{(1)}, t_0\right) + w\left(a_+^{(2)}, c_+^{(1)}, t_0\right) \right) +
\end{aligned}$$

$$\begin{aligned}
& + w \left( a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) w \left( b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \left( w \left( c_+^{(2)}, b_+^{(1)}, t_0 \right) + w \left( a_-^{(2)}, c_+^{(1)}, t_0 \right) \right) + \\
& + w \left( a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) w \left( b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \left( w \left( c_+^{(2)}, b_-^{(1)}, t_0 \right) + w \left( a_+^{(2)}, c_+^{(1)}, t_0 \right) \right) + \\
& + w \left( a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) w \left( b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \left( w \left( c_+^{(2)}, b_-^{(1)}, t_0 \right) + w \left( a_-^{(2)}, c_+^{(1)}, t_0 \right) \right) = \\
& = w \left( a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) \left( w \left( b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) + w \left( b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \right) w \left( a_+^{(2)}, c_+^{(1)}, t_0 \right) + \\
& + w \left( a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) \left( w \left( b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) + w \left( b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \right) w \left( a_-^{(2)}, c_+^{(1)}, t_0 \right) + \\
& + w \left( b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \left( w \left( a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) + w \left( a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) \right) w \left( c_+^{(2)}, b_+^{(1)}, t_0 \right) + \\
& + w \left( b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \left( w \left( a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) + w \left( a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) \right) w \left( c_+^{(2)}, b_-^{(1)}, t_0 \right).
\end{aligned}$$

Thus, due to the existence of an interaction, the Bell inequalities in Wigner form gain time dependence and they can then be written as the following inequality:

$$\begin{aligned}
& w \left( a_+^{(2)}, b_+^{(1)}, t \right) \leq \tag{6} \\
& \leq w \left( a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) \left( w \left( b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) + w \left( b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \right) w \left( a_+^{(2)}, c_+^{(1)}, t_0 \right) + \\
& + w \left( a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) \left( w \left( b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) + w \left( b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \right) w \left( a_-^{(2)}, c_+^{(1)}, t_0 \right) + \\
& + w \left( b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \left( w \left( a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) + w \left( a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) \right) w \left( c_+^{(2)}, b_+^{(1)}, t_0 \right) + \\
& + w \left( b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) \left( w \left( a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) + w \left( a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) \right) w \left( c_+^{(2)}, b_-^{(1)}, t_0 \right).
\end{aligned}$$

We would like to emphasise that (6) is proved on the set of elementary outcomes  $\Omega$ , which is time-independent.

In the absence of any interactions,  $w \left( a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) = w \left( b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) = 0$ , while  $w \left( a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t) \right) = w \left( b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t) \right) = 1$ . Hence (6) devolves to (3), as it should from the physical point of view. The inequality (3) is in turn equivalent to the time-independent inequality (4).

The time-dependent inequality (6) is the main result of this paper. In the next sections we will show how it can be applied to some real experimental situations. We will demonstrate some advantages of the time-dependent inequality (6) over the static inequality (4).

### III. NRQM: ANTICORRELATED SPINS IN AN EXTERNAL MAGNETIC FIELD

Let us consider the following example.

Let a pseudoscalar particle at rest decay to a positron (index “1”) and an electron (index “2”). It is easy to show (see e.g. [12]), that the  $e^+e^-$  pair is in the state with zero momentum and spin. Let us suppose that at the time  $t_0 = 0$ , the spins of the electron and positron are fully anticorrelated along the axis  $z$ . Then the spin wave function of the  $e^+e^-$  pair at  $t = t_0 = 0$  may be written as:

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(2)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(1)} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(2)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(1)} \right]. \quad (7)$$

Let us suppose that the electron-positron system is embedded in a constant and homogeneous magnetic field  $\vec{\mathcal{H}}$ , which is directed along the axis  $y$ . Let us measure the spin projections in the  $(x, z)$  plane on three non-parallel axes  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$ . Also let us require the particles to propagate along the  $y$  axis to prevent rotation of these charged particles in the magnetic field. The spins of the fermions will begin to precess around the  $y$  axis. Let us consider this precession for particles which have a definite spin projection onto the direction  $\vec{a}$ . In a spherical coordinate system at time  $t = t_0 = 0$ , the corresponding electron and positron wave functions are:

$$|\frac{1}{2}, a_+^{(i)}\rangle = \begin{pmatrix} \cos \theta_a/2 \\ \sin \theta_a/2 \end{pmatrix} \quad \text{and} \quad |\frac{1}{2}, a_-^{(i)}\rangle = \begin{pmatrix} -\sin \theta_a/2 \\ \cos \theta_a/2 \end{pmatrix},$$

where  $i = \{1, 2\}$ . The wave function of the electron at an arbitrary time is:

$$|\psi_{a+}^{(2)}(t)\rangle = \begin{pmatrix} \cos(\omega t + \theta_a/2) \\ \sin(\omega t + \theta_a/2) \end{pmatrix}^{(2)} \quad \text{and} \quad |\psi_{a-}^{(2)}(t)\rangle = \begin{pmatrix} -\sin(\omega t + \theta_a/2) \\ \cos(\omega t + \theta_a/2) \end{pmatrix}^{(2)}, \quad (8)$$

and for the positron wave function:

$$|\psi_{a+}^{(1)}(t)\rangle = \begin{pmatrix} \cos(\omega t - \theta_a/2) \\ -\sin(\omega t - \theta_a/2) \end{pmatrix}^{(1)} \quad \text{and} \quad |\psi_{a-}^{(1)}(t)\rangle = \begin{pmatrix} \sin(\omega t - \theta_a/2) \\ \cos(\omega t - \theta_a/2) \end{pmatrix}^{(1)}, \quad (9)$$

where  $\omega = \frac{e\mathcal{H}}{2m_e c}$  is the Larmor precession frequency of the electron.

Taking into account the initial condition (7) and the electron and positron wave functions (8), (9) in the magnetic field, and assuming  $\theta_z = 0$ , we obtain the wave function of the  $e^+e^-$  pair for an arbitrary time  $t$ :

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}^{(2)} \begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \end{pmatrix}^{(1)} - \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}^{(2)} \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{pmatrix}^{(1)} \right].$$



Now let us compute all the probabilities that entering the inequality (6):

$$\begin{aligned}
w(a_+^{(2)}, b_+^{(1)}, t) &= \left| \langle \frac{1}{2}, a_+^{(2)} | \langle \frac{1}{2}, b_+^{(1)} | \Psi(t) \rangle \right|^2 = \frac{1}{2} \sin^2 \left( \frac{\theta_{ba}}{2} + 2\omega t \right), \\
w(a_-^{(2)}, c_+^{(1)}, t) &= \left| \langle \frac{1}{2}, a_-^{(2)} | \langle \frac{1}{2}, c_+^{(1)} | \Psi(t) \rangle \right|^2 = \frac{1}{2} \cos^2 \left( \frac{\theta_{ca}}{2} + 2\omega t \right), \\
w(c_+^{(2)}, b_-^{(1)}, t) &= \left| \langle \frac{1}{2}, c_+^{(2)} | \langle \frac{1}{2}, b_-^{(1)} | \Psi(t) \rangle \right|^2 = \frac{1}{2} \cos^2 \left( \frac{\theta_{cb}}{2} - 2\omega t \right).
\end{aligned} \tag{10}$$

Then:

$$\begin{aligned}
w(a_+^{(2)}(t_0=0) \rightarrow a_+^{(2)}(t)) &= \left| \langle \psi_{a+}^{(2)}(t) | \frac{1}{2}, a_+^{(2)} \rangle \right|^2 = \cos^2(\omega t), \\
w(a_-^{(2)}(t_0=0) \rightarrow a_+^{(2)}(t)) &= \left| \langle \psi_{a-}^{(2)}(t) | \frac{1}{2}, a_+^{(2)} \rangle \right|^2 = \sin^2(\omega t), \\
w(b_+^{(1)}(t_0=0) \rightarrow b_+^{(1)}(t)) &= \left| \langle \psi_{b+}^{(1)}(t) | \frac{1}{2}, b_+^{(1)} \rangle \right|^2 = \cos^2(\omega t), \\
w(b_-^{(1)}(t_0=0) \rightarrow b_+^{(1)}(t)) &= \left| \langle \psi_{b-}^{(1)}(t) | \frac{1}{2}, b_+^{(1)} \rangle \right|^2 = \sin^2(\omega t).
\end{aligned} \tag{11}$$

Substituting (10) and (11) into (6), we obtain the Bell inequality in Wigner form for our model:

$$\sin^2 \left( \frac{\theta_{ba}}{2} + 2\omega t \right) \leq 2 \sin^2(\omega t) + \cos(2\omega t) \left[ \sin^2 \left( \frac{\theta_{ca}}{2} \right) + \sin^2 \left( \frac{\theta_{bc}}{2} \right) \right]. \tag{12}$$

Note that if at the initial time  $t_0 = 0$  the spins of the electron and positron are correlated along any arbitrary direction  $\vec{n}$ , instead of the  $z$  axis, then the derivation of (12) becomes more complex, but ultimately its structure does not change. This fact might not be obvious initially, because of the unique direction in the example which is determined by the direction of the magnetic field  $\vec{\mathcal{H}}$ .

Let us estimate the level of possible violation of inequality (12) assuming one is free to choose the directions of the spin projections and the magnetic field strength. Before we consider the general case, let us examine two important particular cases.

The first case: let  $\omega t = n\pi$ , where  $n = 0, 1, 2 \dots$ . Then (12) becomes the classic time-independent inequality

$$\sin^2 \left( \frac{\theta_{ba}}{2} \right) \leq \sin^2 \left( \frac{\theta_{ca}}{2} \right) + \sin^2 \left( \frac{\theta_{bc}}{2} \right),$$

for which the maximum of violation is reached if  $\theta_{bc} = \theta_{ca} = \frac{\pi}{3}$  and  $\theta_{ba} = \theta_{bc} + \theta_{ca} = \frac{2\pi}{3}$ . With these values we end up with a false inequality  $\frac{3}{4} \leq \frac{1}{2}$ .

The second case appears if  $\omega t = \frac{\pi}{2} + \pi n$ . Then (12) turns into inequality

$$\sin^2\left(\frac{\theta_{ba}}{2}\right) + \sin^2\left(\frac{\theta_{ca}}{2}\right) + \sin^2\left(\frac{\theta_{bc}}{2}\right) \leq 2,$$

for which the maximum of violation  $\left(\frac{9}{4} \leq 2 \text{ or } \frac{1}{4} \leq 0\right)$  is reached with  $\theta_{bc} = \theta_{ca} = \frac{2\pi}{3}$  and  $\theta_{ba} = \theta_{bc} + \theta_{ca} = \frac{4\pi}{3}$ , i.e. with the most symmetric configuration of axes  $\vec{a}$ ,  $\vec{b}$ , and  $\vec{c}$  in the plane  $(x, z)$ . A similar inequality may be obtained for free correlated particles, see e.g. [12].

It is obvious that in both the above cases, the inequality (12) does not have any advantage over the classic inequality (4). Is it possible to violate (12) more than (4) using the magnetic field?

Let us choose the angle  $\theta$  so that  $\cos(2\theta) = \frac{1}{4}$  (i.e.,  $\theta \approx 37, 8^\circ$ ), and let the angles between the spin projections and the magnetic field direction satisfy the following:  $\theta_{bc} = \theta_{ca} = \theta$ ,  $\theta_{ba} = \theta_{bc} + \theta_{ca} = 2\theta$ , and  $\omega t = \frac{\theta}{2} + \pi n$ . In this case the maximal violation of (12) becomes  $\frac{9}{16} \leq 0$ . This violation exceeds the violation of (4) by more than a factor of two.

This result is a consequence of the new degree of freedom provided by the magnetic field.

Note, that every real measurement device has a finite time resolution  $\Delta t$ . To take this into account, let us average (12) using operator

$$I_\delta[f(\tau)] = \frac{1}{2\delta} \int_{T-\delta}^{T+\delta} f(\tau) d\tau,$$

where  $2\delta = \omega\Delta t$  is the dimensionless characteristic resolution of the macro-device,  $\tau = \omega t$  is a dimensionless time parameter. The  $T$  is a dimensionless “common time.” The operator  $I_\delta[f(\tau)]$  affects the summands of (12) as follows:

$$\begin{aligned} I_\delta \left[ \sin^2 \left( \frac{\theta_{ba}}{2} + 2\tau \right) \right] &= \frac{1}{2} \left( 1 - \text{sinc}(4\delta) \right) + \text{sinc}(4\delta) \sin^2 \left( \frac{\theta_{ba}}{2} + 2T \right); \\ I_\delta \left[ \sin^2(\tau) \right] &= \frac{1}{2} \left( 1 - \text{sinc}(2\delta) \right) + \text{sinc}(2\delta) \sin^2(T); \\ I_\delta [\cos(2\tau)] &= \text{sinc}(2\delta) \cos(2T), \end{aligned}$$

where the cardinal sine  $\text{sinc}(x) = \sin(x)/x$ . Substituting these expressions into (12), we obtain an inequality which accounts for the finite time resolution of a macro-device:

$$\begin{aligned} \text{sinc}(2\delta) \left[ \cos(2\delta) \left( \sin^2 \left( \frac{\theta_{ba}}{2} + 2T \right) - \frac{1}{2} \right) + 1 - 2 \sin^2(T) - \right. \\ \left. \cos(2T) \left( \sin^2 \left( \frac{\theta_{ca}}{2} \right) + \sin^2 \left( \frac{\theta_{bc}}{2} \right) \right) \right] \leq \frac{1}{2}. \end{aligned} \quad (13)$$

For an ideal ( $\delta \rightarrow 0$ ) macro-device this inequality becomes inequality (12). If the time resolution of the device substantially exceeds the precession period ( $\delta \gg 1$ ), then the inequality (13) turns into the trivial expression  $0 \leq 1/2$ .

Let us apply 13 to the case of maximum violation of the inequality (12) when  $\cos(2\theta) = \frac{1}{4}$ . Then 13 goes into the inequality

$$K(\delta) = \text{sinc}(2\delta) \left( 0.44 \cos(2\delta) + 0.62 \right) - \frac{1}{2} \leq 0. \quad (14)$$

This inequality depends only on the dimensionless time resolution of the macro-device. The function  $K(\delta)$  is presented in Fig. III. One can see that to experimentally resolve the violation of (12), it is necessary to have  $\delta \lesssim 0.85$ . Then the time resolution of the macro-device may be estimated as

$$\Delta t \lesssim \frac{1.7}{\omega}.$$

Considering a magnetic field  $\mathcal{H} = 1$  T, one can get for an electron  $\Delta t \lesssim 10^{-13}$  s, and for a proton  $\Delta t \lesssim 10^{-7}$  s. These time thresholds might be increased by decreasing the magnetic field strength.

#### IV. TIME-DEPENDENT BELL INEQUALITIES IN WIGNER FORM FOR OSCILLATIONS OF NEUTRAL PSEUDOSCALAR MESONS

Despite the fact that in Section II we used spin projections of fermions, the inequality (6) is correct for any three dichotomic observables. The only requirement for tests of Bohr's principle of complementarity is for corresponding operators not to commute with each other.

To demonstrate the distinction between the inequalities (6) and (4), let us consider as an example the oscillations of neutral  $B$ -mesons, where the static inequalities (4) are never violated, while the dynamical inequalities (6) are violated almost always.

The main idea of static Bell inequalities in Wigner form was introduced in [20, 23, 24] and was further developed in [26, 27, 29]. Its gist is to select three distinct "directions" in the system of neutral  $B$ -mesons, whose operators do not commute with each other. The first "direction" is the  $B$ -meson flavour, i.e. the projection onto the states  $|B\rangle = |\bar{b}q\rangle$  and  $|\bar{B}\rangle = |b\bar{q}\rangle$ , where  $q = \{d, s\}$ . Let us define the operators of charge ( $\hat{C}$ ) and space ( $\hat{P}$ ) conjugation for these states as

$$\hat{C}\hat{P}|B\rangle = e^{i\alpha}|\bar{B}\rangle \quad \text{and} \quad \hat{C}\hat{P}|\bar{B}\rangle = e^{-i\alpha}|B\rangle,$$

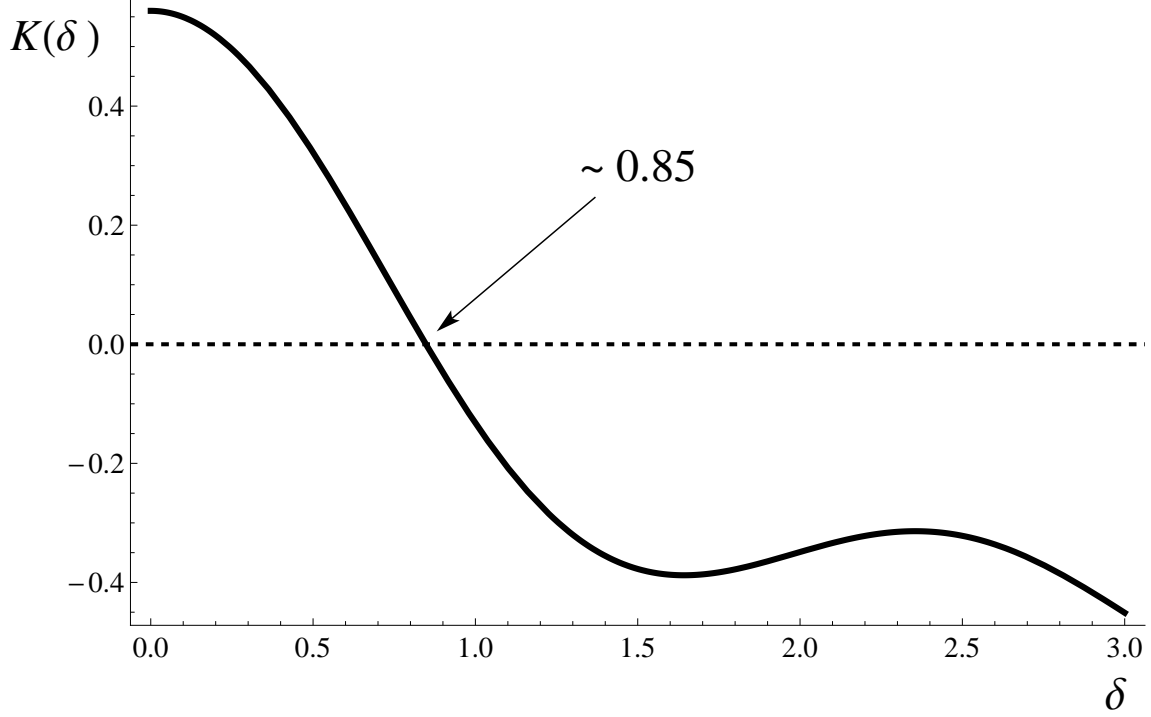


FIG. 1: Violation rate of the inequality 13 versus the time resolution.

where  $\alpha$  is a non-physical arbitrary phase of the  $CP$ -violation. This phase should not appear in any experimentally testable inequalities.

The second “direction” is the states with a definite value of  $CP$ , i.e. the states

$$|B_1\rangle = \frac{1}{\sqrt{2}} (|B\rangle - e^{i\alpha} |\bar{B}\rangle), \quad |B_2\rangle = \frac{1}{\sqrt{2}} (|B\rangle + e^{i\alpha} |\bar{B}\rangle),$$

with negative and positive  $CP$  accordingly.

The third “direction” is defined by the mass and lifetime eigenstates:

$$|B_L\rangle = p|B\rangle + q|\bar{B}\rangle \quad \text{and} \quad |B_H\rangle = p|B\rangle - q|\bar{B}\rangle.$$

The latter two states are the eigenvectors of a non-Hermitian Hamiltonian, with eigenvalues of  $E_L = m_L - i\Gamma_L/2$  and  $E_H = m_H - i\Gamma_H/2$  respectively (we use  $\hbar = c = 1$ ). These two states are not orthogonal to each other. The complex coefficients  $p$  and  $q$  are normalized in the standard way:

$$|p|^2 + |q|^2 = |\tilde{p}|^2 + |q|^2 = 1,$$

where  $\tilde{p} = pe^{i\alpha}$ . The decay  $\Upsilon(4S) \rightarrow B\bar{B}$  defines the wave function of the  $B\bar{B}$ -system in the flavour space at  $t = t_0$ :

$$|\Psi(t_0)\rangle = \frac{1}{\sqrt{2}} \left( |B\rangle^{(2)} |\bar{B}\rangle^{(1)} - |\bar{B}\rangle^{(2)} |B\rangle^{(1)} \right), \quad (15)$$

which is identical to the wave function (7) in spin space. In an experiment one can distinguish between the  $B$ -mesons by their direction in the detector (this procedure is detailed in [23, 26, 27]).

Let us briefly overview the case of the time-independent inequalities. Here we follow the logic of [27]. Let us correspond:  $a_+ \rightarrow B_1$ ,  $a_- \rightarrow B_2$ ,  $b_+ \rightarrow \bar{B}$ ,  $b_- \rightarrow B$ ,  $c_+ \rightarrow B_H$  and  $c_- \rightarrow B_L$ . Then the classic inequality (4) becomes:

$$w(B_1^{(2)}, \bar{B}^{(1)}, t_0) \leq w(B_1^{(2)}, B_H^{(1)}, t_0) + w(B_H^{(2)}, \bar{B}^{(1)}, t_0). \quad (16)$$

Using the initial condition (15) one finds that:

$$\begin{aligned} w(B_1^{(2)}, \bar{B}^{(1)}, t_0) &= \left| \langle B_1^{(2)} | \langle \bar{B}^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} = \frac{1}{4} (|\tilde{p}|^2 + |q|^2); \\ w(B_1^{(2)}, B^{(1)}, t_0) &= \left| \langle B_1^{(2)} | \langle B^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} = \frac{1}{4} (|\tilde{p}|^2 + |q|^2); \\ w(B_1^{(2)}, B_H^{(1)}, t_0) &= \left| \langle B_1^{(2)} | \langle \bar{B}_H^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} |\tilde{p} - q|^2; \\ w(B_2^{(2)}, B_H^{(1)}, t_0) &= \left| \langle B_2^{(2)} | \langle \bar{B}_H^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{4} |\tilde{p} + q|^2; \\ w(B_H^{(2)}, \bar{B}^{(1)}, t_0) &= \left| \langle B_H^{(2)} | \langle \bar{B}^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{2} |\tilde{p}|^2, \\ w(B_H^{(2)}, B^{(1)}, t_0) &= \left| \langle B_H^{(2)} | \langle B^{(1)} | \Psi(t_0) \rangle \right|^2 = \frac{1}{2} |q|^2. \end{aligned} \quad (17)$$

Substituting the probabilities (17) into (16) leads to:

$$|q|^2 - |\tilde{p}|^2 \leq |\tilde{p} - q|^2, \quad (18)$$

which is identical to the inequality (16) from [23] for the oscillations of neutral  $K$ -mesons if one redefines  $p$  and  $q$  through the  $CP$ -violation parameter  $\varepsilon$ . As there is not direct dependence between the projections of the states onto various directions, one can suppose that  $b_+ \rightarrow B$  and  $b_- \rightarrow \bar{B}$ . Then (4) becomes

$$w(B_1^{(2)}, B^{(1)}, t_0) \leq w(B_1^{(2)}, B_H^{(1)}, t_0) + w(B_H^{(2)}, B^{(1)}, t_0), \quad (19)$$

and consequently, taking into account (17):

$$|\tilde{p}|^2 - |q|^2 \leq |\tilde{p} - q|^2. \quad (20)$$

The inequalities (18) and (20) may be uniformly written as:

$$||\tilde{p}|^2 - |q|^2| \leq |\tilde{p} - q|^2. \quad (21)$$

Relation (21) is never violated, as follows from the triangle inequality for complex numbers. The equality is reached when  $|\tilde{p}| = |q|$  or when  $|p| = |q|$ . All possible re-definitions of “directions” and projections of the  $B$ -meson states upon them lead to inequality (21). In [26, 27] it is supposed that the inequality similar to (18) may be violated by a special choice of the non-physical phase  $\alpha$ . But it is obvious that the inequalities (18), (20), and (21) do not depend on  $\alpha$ , as should be the case for any experimentally testable theory. Hence the static inequalities (4), written in terms of the oscillation parameters of neutral  $B$ -mesons (16), (19), are never violated, and do not allow testing of the complementarity principle.

Let us now consider the time-dependent inequalities (6), written for a system of neutral  $B$ -mesons. Note that for the derivation of (6) the probability normalisation was never used. Hence (6) is valid for decays in which the normalisation of the state vectors depends on time. Again set  $a_+ \rightarrow B_1$ ,  $a_- \rightarrow B_2$ ,  $b_+ \rightarrow \bar{B}$ ,  $b_- \rightarrow B$ ,  $c_+ \rightarrow B_H$  and  $c_- \rightarrow B_L$ . The time evolution of the states  $|B_L\rangle$  and  $|B_H\rangle$  is trivial:

$$|B_L(t)\rangle = e^{-im_L t - \Gamma_L t/2} |B_L\rangle, \quad |B_H(t)\rangle = e^{-im_H t - \Gamma_H t/2} |B_H\rangle.$$

This defines the evolution of the states  $|B(t)\rangle$  and  $|\bar{B}(t)\rangle$  as follows:

$$\begin{cases} |B(t)\rangle = g_+(t)|B\rangle - \frac{q}{p}g_-(t)|\bar{B}\rangle \\ |\bar{B}(t)\rangle = -\frac{p}{q}g_-(t)|B\rangle + g_+(t)|\bar{B}\rangle \end{cases}.$$

Now one can derive the evolution of the state  $|B_1(t)\rangle$  as:

$$|B_1(t)\rangle = \frac{1}{\sqrt{2}} \left( \left( g_+(t) + \frac{p}{q} e^{i\alpha} g_-(t) \right) |B\rangle - \left( g_+(t) e^{i\alpha} + \frac{q}{p} g_-(t) \right) |\bar{B}\rangle \right),$$

where  $g_{\pm}(t) = \frac{1}{2} (e^{-iE_H t} \pm e^{-iE_L t})$ . Functions  $g_{\pm}(t)$  satisfy the conditions:

$$\begin{aligned} |g_{\pm}(t)|^2 &= \frac{e^{-\Gamma t}}{2} \left( \text{ch} \left( \frac{\Delta\Gamma t}{2} \right) \pm \cos(\Delta m t) \right) \text{ and} \\ g_+(t)g_-^*(t) &= \frac{e^{-\Gamma t}}{2} \left( -\text{sh} \left( \frac{\Delta\Gamma t}{2} \right) + i \sin(\Delta m t) \right), \end{aligned}$$

where  $\Gamma = (\Gamma_H + \Gamma_L)/2$ ,  $\Delta\Gamma = \Gamma_H - \Gamma_L$ , and  $\Delta m = m_H - m_L$ . Taking into account initial condition (15), it is possible to write for the wave function of the  $B\bar{B}$ -pair at an arbitrary

moment of time:

$$|\Psi(t)\rangle = e^{-i(m_H+m_L)t} e^{-\Gamma t} |\Psi(t_0)\rangle. \quad (22)$$

For the neutral  $B$ -mesons,  $\left(\frac{q}{p}\right)^2 = e^{2i\alpha}$  [35]. The experimental value for this ratio is [35]:

$$\left|\frac{q}{p}\right| = 1.0017 \pm 0.0017.$$

Considering the normalisation at  $t = t_0$ , we can let  $|p|^2 \approx |q|^2 \approx 1/2$ . Subsequent calculations will be performed for the case of  $q/p = e^{i\alpha}$ . Then:

$$\begin{aligned} w(B_1^{(2)}(0) \rightarrow B_1^{(2)}(t)) &= |\langle B_1(t) | B_1 \rangle|^2 = e^{-\Gamma t} e^{-\Delta\Gamma t/2}; \\ w(B_2^{(2)}(0) \rightarrow B_1^{(2)}(t)) &= |\langle B_1(t) | B_2 \rangle|^2 = 0; \\ w(\bar{B}^{(1)}(0) \rightarrow \bar{B}^{(1)}(t)) &= |\langle \bar{B}(t) | \bar{B} \rangle|^2 = |g_+(t)|^2; \\ w(B^{(1)}(0) \rightarrow \bar{B}^{(1)}(t)) &= |\langle \bar{B}(t) | B \rangle|^2 = |g_-(t)|^2; \\ w(B_1^{(2)}, \bar{B}^{(1)}, t) &= \left| \langle B_1^{(2)} | \langle \bar{B}^{(1)} | \Psi(t) \rangle \right|^2 = \frac{1}{4} e^{-2\Gamma t}. \end{aligned} \quad (23)$$

The substitution of (17) and (23) into (6) results in the following time-dependent inequality:

$$1 \leq e^{-\Delta\Gamma t}. \quad (24)$$

If  $\Delta\Gamma = \Gamma_H - \Gamma_L > 0$ , then the inequality (24) is violated for any  $t > 0$ . Note that (24) does not contradict the static inequality (21), because with  $q/p = e^{i\alpha}$  the latter also reverts to equality.

While the static Bell inequality in Wigner form is never violated in the example, regardless of the choices of  $p$  and  $q$ , the dynamical inequality is always violated for  $q/p = e^{i\alpha}$ , if  $\Delta\Gamma = \Gamma_H - \Gamma_L > 0$ . The latter condition is proven experimentally for the neutral  $B_{d,s}$ -mesons [35].

## V. QFT: BELL INEQUALITIES AT THE LEADING ORDER

In the framework of NRQM, the violation of the time-dependent Bell inequalities in Wigner form (6), like the violation of the static inequalities (4), is determined by its non-locality. That is, this violation in NRQM means one of the following:

- a) NRQM is non-local, but the spin projections onto non-parallel directions may simultaneously be the elements of the physical reality;
- b) NRQM is non-local, but Bohr's complementarity principle is valid, i.e. the spin projections depends on the configuration of a macro-device.

Hence, in order to test the complementarity principle alone, it is necessary to exclude non-locality. This is possible if one considers (6) in the framework of QFT, which is local by construction. In QFT the experimental proof of a violation of (6) will mean the validity of the complementarity principle.

Standard methods in QFT only allow calculation of the time-dependent decay probability  $W(t) = \partial w(t)/\partial t$  with perturbative theory. As an example let us consider a decay of a free neutral pseudoscalar particle  $P$  to a fermion-antifermion pair  $f^+ f^-$ . Let antifermion  $f^+$  to have the index "1", and fermion  $f^-$  to have the index "2". The Hamiltonian of the decay is defined by expression (2), where  $f(x)$  and  $\bar{f}(x)$  are fermionic fields, and  $\varphi(x)$  is a pseudoscalar field. Let the pseudoscalar particle have mass  $M$ , and let the fermionic masses to be negligible (this assumption will not affect the final result but will simplify the calculations). The decay width  $\Gamma_0$  when  $t_0 \rightarrow -\infty$  or  $t \rightarrow +\infty$  is equal to  $\Gamma_0 = \frac{g^2 M}{8\pi}$ . However the probabilities in (6) are defined for finite times  $t$  and  $t_0$ . Let us calculate them.

Using the technique of calculations in QFT for finite times [36–38], one obtains the following expression for the probability of the decay  $P \rightarrow f^+ f^-$ :

$$W^{(1)}(a_+^{(2)}, b_+^{(1)}, \tau) = \frac{\Gamma_0}{2} \left( 1 + \frac{\text{si}(M\tau)}{\pi} + \frac{\sin(M\tau)}{\pi(M\tau)^2} + \frac{\cos(M\tau t)}{\pi(M\tau t)} \right) \times \sin^2 \frac{\theta_{ab}}{2}, \quad (25)$$

where  $\tau$  is the time of measurement, and  $\text{si}(x)$  is the integral sine, defined as

$$\text{si}(x) = - \int_x^{+\infty} \frac{\sin \zeta}{\zeta} d\zeta.$$

For  $\tau \rightarrow +\infty$ , the expression (25) goes into  $W^{(1)}(a_+^{(2)}, b_+^{(1)}) = \Gamma_0 \times \sin^2 \frac{\theta_{ab}}{2}$ , as it should. For  $M \rightarrow 0$ ,  $W^{(1)}(a_+^{(2)}, b_+^{(1)}, \tau) \rightarrow 0$  because of the phase space reduction. For  $\tau \rightarrow 0$  the expression (25) has a pole for  $\tau$ :

$$W^{(1)}(a_+^{(2)}, b_+^{(1)}, \tau \rightarrow 0) \approx \frac{\Gamma_0}{2} \left( \frac{1}{2} + \frac{2}{\pi(M\tau)} \right) \times \sin^2 \frac{\theta_{ab}}{2}.$$

The existence of the pole when  $\tau \rightarrow 0$  has been discussed in many works [36, 38–40]. It was shown that the pole is not related to the ultraviolet divergence of QFT, and, hence,



can not be removed by usual renormalisation technique. The presence of such “surface” divergencies was noted long ago in [41], where the concept of an interaction in different regions of space-time was clearly stated. The surface divergences are our “penalty” for using the Dirac picture and the approximation of non-interaction in finite time. However in our example, times  $\tau$  are cut off by the time resolution  $\Delta t$  of the detector, which measures properties of the fermions in the final state. It is obvious that  $\Delta t \gg 1/M$ . Hence the pole  $\tau$  (25) is not really essential here.

For the leading order to suffice, the full decay width should be much less than the mass of the decaying particle. Let us suppose that the decay  $P \rightarrow f^+ f^-$  is absolutely dominant. Then the condition for using perturbative theory is:

$$\frac{\Gamma_0}{2} \left( 1 + \frac{\text{si}(M\tau)}{\pi} + \frac{\sin(M\tau)}{\pi(M\tau)^2} + \frac{\cos(M\tau)}{\pi(M\tau)} \right) = W^{(1)}(\tau) \approx \Gamma(\tau) \ll M. \quad (26)$$

If the coupling constant  $g$  is small enough, then it is possible to obtain  $\frac{\Gamma_0}{M} \ll 1$  as small as needed. Hence condition (26) is satisfied for a wide range of the value  $M\tau \gg 1$ .

Now let us return to the inequality (6). Times  $t_0$  and  $t$  should be interpreted as the time intervals of measurement of each of the probabilities. The probabilities of the flip of the spins of the fermion  $w(a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t))$  and antifermion  $w(b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t))$  in the right part of (6) are much smaller by the coupling constant  $g$ . Hence at the leading order  $w(a_-^{(2)}(t_0) \rightarrow a_+^{(2)}(t)) \approx w(b_-^{(1)}(t_0) \rightarrow b_+^{(1)}(t)) \approx 0$ . Analogously, without the spin flip  $w(a_+^{(2)}(t_0) \rightarrow a_+^{(2)}(t)) \approx w(b_+^{(1)}(t_0) \rightarrow b_+^{(1)}(t)) \approx 1$ .

Integration (25) over  $d\tau$  from  $t_i$  to  $t_f$  assuming that  $\frac{M}{\Gamma_0} \gg M\tau \gg 1$ , gives:

$$\begin{aligned} w(a_+^{(2)}, b_+^{(1)}, t_f - t_i) &\approx \int_{t_i}^{t_f} d\tau W^{(1)}(a_+^{(2)}, b_+^{(1)}, \tau) \approx \\ &\approx \frac{\Gamma_0}{2} \sin^2 \frac{\theta_{ab}}{2} \int_{t_i}^{t_f} d\tau = \frac{(t_f - t_i) \Gamma_0}{2} \sin^2 \frac{\theta_{ab}}{2}. \end{aligned} \quad (27)$$

Let the measurement time be  $t_f - t_i = t_0$  on the right side of (25), and on the left side set it to  $t_f - t_i = t$ . Then:

$$\frac{t}{t_0} \sin^2 \left( \frac{\theta_{ba}}{2} \right) \leq \sin^2 \left( \frac{\theta_{ca}}{2} \right) + \sin^2 \left( \frac{\theta_{bc}}{2} \right). \quad (28)$$

This inequality differs from the classic inequality (4) by the ratio  $\frac{t}{t_0} \geq 1$  on the left side. Note that it is correct to divide by  $t_0$ , because  $t_0 \geq \Delta t \gg 0$ . The ratio  $t/t_0$  may exceed unity by a few times, meaning that the area of angular values which violate the inequality is much wider. From the experimental point of view it is more suitable to use the ratio of distances between the polarizers  $L/L_0$  instead of the ratio of times  $t/t_0$ . That is, in experiments measuring the spin projections to  $(\vec{a}, \vec{c})$  and  $(\vec{b}, \vec{c})$ , it should be equal to  $L_0$ , and for  $\vec{a}$  and  $\vec{b}$  it should be  $L, \frac{M}{\Gamma_0} \gg M \{L, L_0\} \gg 1$ .

There are no terms in (28) specific to Hamiltonian (2), meaning this inequality is valid for any other QFT Hamiltonian if it allows anticorrelation (1) in the limits of applicability of the perturbation theory's leading order. In this sense inequality (28) may be considered a universal time-dependent Bell inequality in the QFT in the absence of external fields.

## VI. CONCLUSION

Under the assumption of local realism and using Kolmogorov's probability theory we obtain the time-dependent Bell inequalities in Wigner form (6). These inequalities may be used in non-relativistic quantum mechanics as well as in the quantum field theory, where it is impossible to exclude field interactions. We consider a few examples to which the inequality (6) may be applied. In all cases we demonstrate the extension of the range of parameters for the violation of (6) in comparison to that of the classic inequalities (4).

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